The More, the Less, and the Much More: An Introduction to Łukasiewicz logic, Part 2

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DIDATTICA

- Attrattività
- Sostenibilità
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- Borse di studio
- Dispersione
- Efficacia
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- Occupazione

RICERCA

- Fondi esterni
- Ricerca
- Alta formazione

POSIZIONE	ATENEO	PUNTI
1	Verona	84
2	Trento	84
3	Politecnico di Milano	79
4	Bologna	78
5	Padova	76
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8	Milano Bicocca	73
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10	Politecnico di Torino	73
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16	Macerata	65
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22	Roma Foro Italico	58
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24	Pisa	56

Intermezzo

Ada Lovelace, 1815 – 1852

By now Ada is beginning to feel a little more optimistic about the possibility of applying reasoning to her problem.

Semantics

1. The intended semantics is necessarily informal, and is one of the factors that can inspire the definition of a formal deductive system. In our case, the intended semantics is given by vague predicates/propositions.
2. The formal semantics, on the other hand, is what we usually mean by "semantics" in formal logic: a mathematical construct (logical valuations, Tarskian structures, Kripke frames. . .) that formalises the intended semantics, hopefully leading to a completeness theorem for the formal deductive system.

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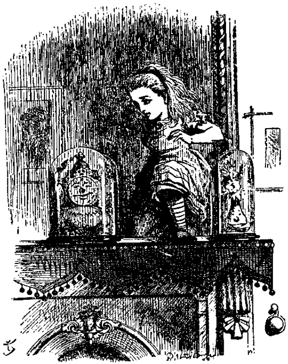
3. Something we mathematical logicians learnt in the last quarter of the 20th century:

Under reasonable assumption (e.g. algebraisability), the formal deductive system canonically induces its own formal semantics.

The Semantics — whatever it turns out to be



$$\begin{tikzcd} \mathsf{Boole}^{\mathrm{op}} \arrow[rr, bend right] & & \mathsf{Stone} \arrow[ll, bend right] \end{tikzcd}$$

Axiom system.

(A0) $\neg(\alpha \rhd \top)$ *Ex falso quodlibet*

(A1) $\alpha \rhd \beta \;\leqslant\; \alpha$ *A fortiori*

(A2) $(\gamma \rhd \alpha) \rhd (\gamma \rhd \beta) \;\leqslant\; \beta \rhd \alpha$ Transitivity of $\rhd$

(A3) $\alpha \rhd (\alpha \rhd \beta) \;\leqslant\; \beta \rhd (\beta \rhd \alpha)$ Conjunction is commutative

(A4) $\alpha \rhd \beta \;\leqslant\; \neg\beta \rhd \neg\alpha$ Contraposition

$$\alpha \wedge \beta \;\equiv\; \alpha \rhd (\alpha \rhd \beta)$$

Deduction rule.

(R1) $\dfrac{\alpha \leqslant \beta \qquad \neg\beta}{\neg\alpha}$ *Vague Modus Tollens.*

Lindenbaum's Equivalence Relation

Formulæ α, β are logically equivalent if $\vdash \alpha \leqslant \beta$ and $\vdash \beta \leqslant \alpha$.
Write $\alpha \equiv \beta$.

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On the quotient set $\frac{\textsc{Form}}{\equiv}$, the connectives induce operations:

- $1 := [\top]_{\equiv}$
- $\neg[\alpha]_{\equiv} := [\neg\alpha]_{\equiv}$
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The algebraic structure $(\frac{\textsc{Form}}{\equiv}, \rhd, \neg, 1)$ is an MV-algebra.

> 'MV-algebra' is short for 'Many-Valued Algebra', *"for lack of a better name."*
>
> (C.C. Chang, 1986).

$$\frac{\text{MV-algebras : Łukasiewicz logic}}{} = \frac{\text{Boolean algebras : Classical logic}}{}$$

MV-algebras are usually presented over the adjoint signature $\oplus$, $\neg$, 0. Here $x \oplus y := \neg((\neg x) \rhd y)$.

Abstractly: $(M, \oplus, \neg, 0)$ is an MV-algebra if $(M, \oplus, 0)$ is a commutative monoid, $\neg\neg x = x$, $1 := \neg 0$ is absorbing for $\oplus$ ($x \oplus 1 = 1$), and, characteristically,

$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x \tag{*}$$

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Any MV-algebra has an underlying distributive lattice bounded below by 0 and above by 1. Joins are given by

$$x \vee y := \neg(\neg x \oplus y) \oplus y$$

Thus, the characteristic law (*) states that joins commute:

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Boolean algebras=Idempotent MV-algebras: $x \oplus x = x$.

Theories in Łukasiewicz logic are as usual: deductively closed sets of formulæ.

A theory is consistent if it fails to contain at least one formula.

A theory is maximal consistent, or just maximal, if it is consistent, and inclusion-maximal with that property.

A theory Θ is prime if it is consistent, and for any α and β either $\Theta \vdash \alpha \leqslant \beta$ or $\Theta \vdash \beta \leqslant \alpha$ holds.

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For an arbitrary set S of formulæ, its deductive closure $S^{\vdash}$ is the intersection of all theories that contain S.

The above terminology generalises to S in the obvious manner, e.g. S is maximal consistent if $S^{\vdash}$ is maximal.

Given a theory Θ, a set of formulæ S is said to axiomatise Θ just in case $S^{\vdash} = \Theta$.

We now restrict attention to the fragment of Łukasiewicz logic over one variable, X.

There are now natural bijections (up to isomorphism) between:

- MV-algebras, and
- Theories in Łukasiewicz logic.

And between:

- Linearly ordered MV-algebras, and
- Prime theories in Łukasiewicz logic.

And between:

- Simple MV-algebras, and
- Maximal theories in Łukasiewicz logic.

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Note

Any formula in Łukasiewicz logic can be evaluated into any MV-algebra, by construction.

Theorem (Essentially O. Hölder, 1901)

If A is a simple MV-algebra, then there is a unique MV-algebraic embedding

$$A \hookrightarrow [0,1].$$

Here, the interval $[0,1] \subseteq \mathbb{R}$ is made into an MV-algebra with neutral element 0 by defining

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Theorem (Chang's completeness theorem, 1959)

The variety of MV-algebras is generated by $[0,1]$.

C.C. Chang, *Trans. of the AMS*, 1959.

Now define a valuation, *tout court*, to be an evaluation of the entire set FORM into MV-algebra $[0,1]$ — or equivalently, into any simple MV-algebra. Write

$$\vDash \alpha$$

if each valuation w satisfies $w(\alpha) = 1$. Then, from Chang's theorem:

Soundness and Completeness Theorem for Ł

For any $\alpha \in \mathrm{FORM}$,

$$\vdash \alpha \quad \text{if, and only if,} \quad \vDash \alpha\,.$$

A. Rose and J. Barkley Rosser, *Trans. of the AMS*, 1958.

Let us consider the *tertium non datur* equation:

$$x \vee \neg x \; = \; 1\,. \tag{$\star$}$$

Then ($\star$) is not an identity over $[0,1]$: the only evaluations into $[0,1]$ that satisfy ($\star$) are $x \mapsto 0$ and $x \mapsto 1$ — the Boolean, or classical, evaluations.

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The boundary of the unit square.

$$X \vee \neg X = 1 \qquad (\star)$$

The boundary of the unit interval.

$$X \vee \neg X \vee Y \vee \neg Y = 1 \qquad (\star\star)$$

The boundary of the unit square.

Rational polyhedra

Leonardo's Truncated Icosahedron

(*Illustration for Luca Pacioli's The Divine Proportion, 1509.*)

We consider finitely presented MV-algebras, *i.e.* those of the form $\mathscr{F}_n/\theta$, with θ a finitely generated congruence (ideal). The assumption on θ is far from immaterial: there is no Hilbert's Basis Theorem for MV-algebras.

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The convex hull of a set $P \subseteq \mathbb{R}^n$, written conv P, is the collection of all convex combinations of elements of P:

$$\operatorname{conv} P = \left\{ \sum_{i=1}^{m} r_i v_i \mid v_i \in P \text{ and } 0 \leqslant r_i \in \mathbb{R} \text{ with } \sum_{i=1}^{m} r_i = 1 \right\}.$$

Such a set is convex if $P = \operatorname{conv} P$.

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The set P is called:

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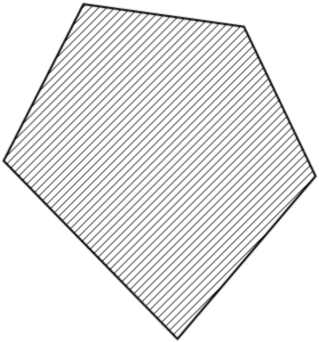
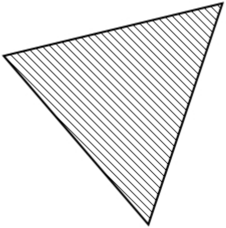
Such a set is convex if $P = \text{conv } P$.

The set P is called:

- a polytope, if there is a finite $F \subseteq \mathbb{R}^n$ with $P = \text{conv } F$;
- a rational polytope, if there is a finite $F \subseteq \mathbb{Q}^n$ with $P = \text{conv } F$.

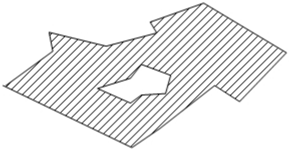


A polytope in $\mathbb{R}^2$.



A polytope in $\mathbb{R}^2$ (*a* simplex).

A (compact) polyhedron in $\mathbb{R}^n$ is a union of finitely many polytopes in $\mathbb{R}^n$.

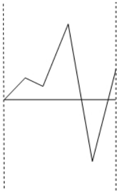


A polyhedron in $\mathbb{R}^2$.

Similarly, a rational polyhedron is a union of finitely many rational polytopes.

Let $P \subseteq \mathbb{R}^n$ be a rational polyhedron. A continuous function $f : P \to \mathbb{R}$ is a $\mathbb{Z}$-map if the following hold.

1. There is a finite set $\{L_1, \ldots, L_m\}$ of affine linear functions $L_i : \mathbb{R}^n \to \mathbb{R}$ such that $f(x) = L_{i_x}(x)$ for some $1 \leqslant i_x \leqslant m$.



A piecewise linear function $[0, 1] \to \mathbb{R}$.

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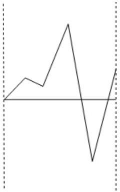
2. Each L_i can be written as a linear polynomial with integer coefficients.



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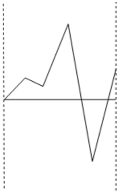

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A map $F\colon P \subseteq \mathbb{R}^n \to Q \subseteq \mathbb{R}^m$ between polyhedra always is of the form $F = (f_1, \ldots, f_m)$, $f_i\colon P \to \mathbb{R}$. Then F is a $\mathbb{Z}$-map if each one of its scalar components f_i is.

Rational polyhedra are precisely the subsets of $\mathbb{R}^n$ that are definable by a term in the language of MV-algebras; and $\mathbb{Z}$-maps are precisely the continuous transformations that are definable by tuples of terms in that language.

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Stone-type duality for finitely presented MV-algebras

The category of finitely presented MV-algebras, and their homomorphisms, is equivalent to the opposite of the category of rational polyhedra, and the $\mathbb{Z}$-maps amongst them.

- V.M. & L. Spada, *Duality, projectivity, and unification in Łukasiewicz logic and MV-algebras*, Annals of Pure and Applied Logic, 2012.

$$\begin{tikzcd}
\mathsf{MV}_{\mathrm{fp}} \arrow[rr, bend right] & & \mathsf{Poly}_{\mathbb{Q}}^{\mathrm{op}} \arrow[ll, bend right]
\end{tikzcd}$$

From MV-algebras to rational polyhedra: Given $\mathscr{F}_n/\langle\tau(x_1,\ldots,x_n)\rangle$, the associated rational polyhedron $\mathbb{V}(\tau)$ is the set of n-tuples $(r_1,\ldots,r_n)\in[0,1]^n$ such that $\tau(r_1,\ldots,r_n)=0$ in $[0,1]$.

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From rational polyhedra to MV-algebras: Given $P \subseteq \mathbb{R}^n$, the collection $\nabla(P)$ of all $\mathbb{Z}$-maps $P \to [0,1]$ is a (finitely presentable) MV-algebra under the pointwise operation inherited from $[0,1]$.

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Example. If $\tau(x_1,\ldots,x_n)$ is identically equal to 0 in any MV-algebra, then it generates the trivial ideal $\{0\}$. In this case, $\mathscr{F}_n/\langle\tau\rangle=\mathscr{F}_n$, and $\mathbb{V}(\tau)=[0,1]^n$. Hence the duals of free algebras are the unit cubes.

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Remark. The subspace $\mathbb{V}(\tau)\subseteq[0,1]^n$ homeomorphic to the maximal spectral space of $\mathscr{F}_n/\langle\tau\rangle$, topologised by the (analogue of) the Zariski topology. The MV-algebra $\nabla(P)$ is the exact analogue for rational polyhedra of the coordinate ring of an affine algebraic variety.

The syntax-semantics dictionary.

Algebra, or Syntax.	**Geometry, or Semantics.**
F.p. algebra	Rational polyhedron
Homomorphism	$\mathbb{Z}$-map
F.p. subalgebra	Continuous image by $\mathbb{Z}$-map
F.p. quotient algebra	Rational subpolyhedron
F.p. projective algebra	Retract of cube by $\mathbb{Z}$-maps
Free n-gen. algebra	$[0,1]^n$
Maximal congruence	Point of rational polyhedron
Intersection of maximal cong.	Closed subset of rational polyhedron
Finite product $A \times B$	Finite disjoint union
$\vdots$	$\vdots$

Numbers out of Formulæ

Otto Hölder, 1859–1937.

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In Łukasiewicz logic:

> The degree of truth attached to X (in a given possible world, i.e. valuation) is the set of answers to a tree of yes/no questions.

$$\boxed{\vdash \alpha\ ?}$$

$$\begin{aligned} L_{0,1} &= \neg X \\ R_{0,1} &= X \end{aligned}$$

$$\begin{aligned} L_{1,1} &= L_{0,1} \rhd R_{0,1} \\ R_{1,1} &= R_{0,1} \end{aligned} \qquad \begin{aligned} L_{1,2} &= L_{0,1} \\ R_{1,2} &= R_{0,1} \rhd L_{0,1} \end{aligned}$$

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$$\begin{matrix} L_{3,1} & L_{3,2} & L_{3,3} & L_{3,4} & L_{3,5} & L_{3,6} & L_{3,7} & L_{3,8} \\ R_{3,1} & R_{3,2} & R_{3,3} & R_{3,4} & R_{3,5} & R_{3,6} & R_{3,7} & R_{3,8} \\ \vdots & & & & & & & \vdots \end{matrix}$$

The Yes/No Questions.

It turns out that the intrinsic degree of truth we can attach to a formula $\alpha(X)$ is a possibly infinite branch of the tree, and that these branches are in bijection (and thus encode) with sequences of questions as mentioned above.

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Let us encode a branch of the tree into a (finite or infinite) sequence of left-right steps downward from the root, as in

$$\mathrm{lrllllrrrlrlrlr}\ldots$$

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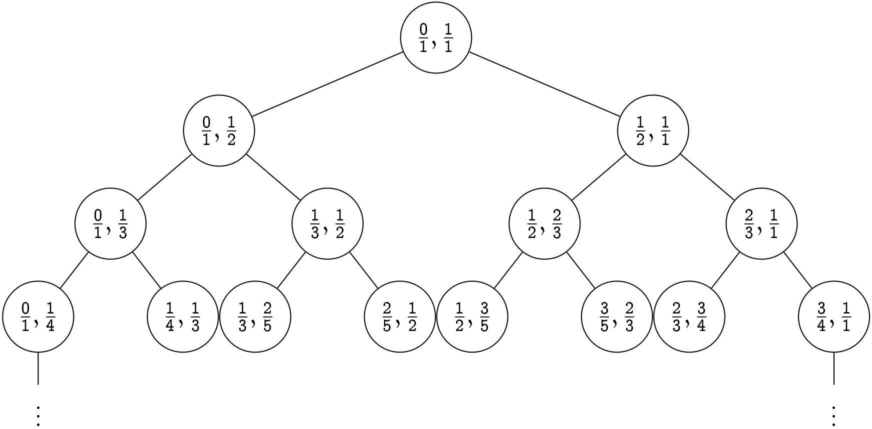
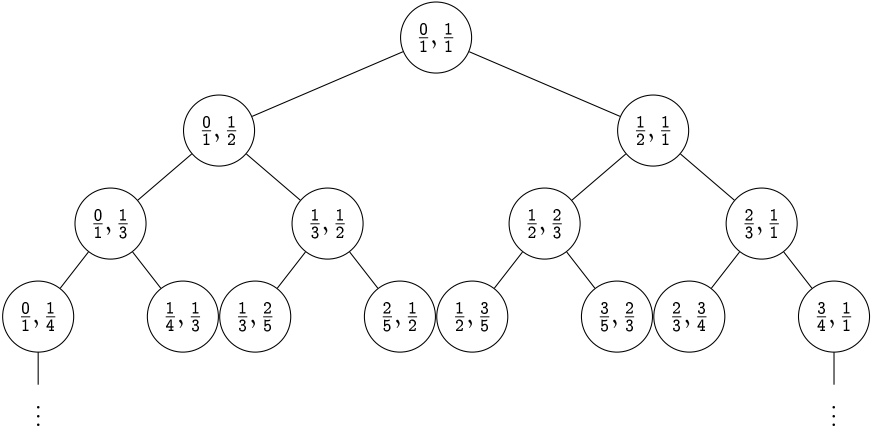
- The finite sequences: Classify finitely axiomatisable theories, or equivalently the elements of $[0,1] \cap \mathbb{Q}$.
- The infinite sequences that are not definitively constant: Classify maximal, non-finitely-axiomatisable theories, or equivalently the elements of $[0,1] \setminus \mathbb{Q}$.
- The infinite definitively constant sequences: Classify prime, non-maximal theories, or equivalently the elements of $[0,1] \cap \mathbb{Q}$, plus or minus a linear infinitesimal.



The Farey tree.



Cauchy's Theorem. Every rational number in $(0, 1)$ occurs, automatically in reduced form, as the mediant of the numbers in some node of the Farey tree exactly once. (The mediant of $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{a+c}{b+d}$.)

$$\begin{aligned} L_{0,1} &= \neg X \\ R_{0,1} &= X \end{aligned}$$

$$\begin{aligned} L_{1,1} &= L_{0,1} \rhd R_{0,1} \\ R_{1,1} &= R_{0,1} \end{aligned} \qquad \begin{aligned} L_{1,2} &= L_{0,1} \\ R_{1,2} &= R_{0,1} \rhd L_{0,1} \end{aligned}$$

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Thm. There are natural bijections between the finitely axiomatisable maximal consistent theories in Ł over 1 variable X, the nodes of the Farey tree together with $\{0, 1\}$, and the rational numbers in $[0, 1]$.

Taking stock

To attach a degree of truth to a formula (=vague proposition such as "VM is tall") in Łukasiewiciz logic mean to consider that formula subject to a prime consistent theory. In the special case that the theory is maximal, the degree of truth can be canonically identified with a unique real number in $[0, 1]$.

Taking stock

To attach a degree of truth to a formula (=vague proposition such as "VM is tall") in Łukasiewiciz logic mean to consider that formula subject to a prime consistent theory. In the special case that the theory is maximal, the degree of truth can be canonically identified with a unique real number in $[0, 1]$.

The prime theory has, moreover, a canonical — though not recursively computable! — axiomatisation whose interpretation in the intended semantics of vague propositions yield the intuitive content of what it means, for example, to say that "VM is tall" is true to degree $\frac{1}{2}$ — or $\frac{\pi}{5\sqrt{2}}$, for that matter.

Epilogue

42

A note on “Very” and “Somewhat”

The monoidal “conjunction”

$$\alpha \odot \beta \equiv \alpha \rhd \neg\beta$$

Iterations of $\odot$ express weakenings in the intended semantics. If $X :=$ “VM is tall”, then $X \odot X :=$ Very (“VM is tall”), i.e. “It is very true that VM is tall”.

Caution

The binary monoidal connective $\odot$, adjoint to $\rightarrow$, is not interpretable as a conjunction in the intended semantics.

A note on "Very" and "Somewhat"

The monoidal "disjunction"

$$\alpha \oplus \beta \equiv \neg(\neg\alpha \rhd \beta)$$

Iterations of $\oplus$ express weakenings in the intended semantics. If $X :=$ "VM is tall", then $X \oplus X :=$ Somewhat ("VM is tall"), i.e. "It is somewhat true that VM is tall".

Caution

The binary monoidal connective $\oplus$, adjoint to $\ominus$, is not interpretable as a disjunction in the intended semantics.

DIDATTICA

- Attrattività
- Sostenibilità
- Stage
- Mobilità Internazionale
- Borse di studio
- Dispersione
- Efficacia
- Soddisfazione
- Occupazione

RICERCA

- Fondi esterni
- Ricerca
- Alta formazione

POSIZIONE	ATENEO	PUNTI
1	Verona	84
2	Trento	84
3	Politecnico di Milano	79
4	Bologna	78
5	Padova	76
6	Politecnica delle Marche	75
7	Venezia Ca' Foscari	73
8	Milano Bicocca	73
9	Siena	73
10	Politecnico di Torino	73
11	Pavia	72
12	Piemonte Orientale	71
13	Milano Statale	70
14	Ferrara	68
15	Udine	66
16	Macerata	65
17	Firenze	63
18	Viterbo	62
19	Modena e Reggio Emilia	61
20	Venezia Iuav	60
21	Torino	59
22	Roma Foro Italico	58
23	Salerno	58
24	Pisa	56

Thank you for your attention.